Lecture Notes, January 7 & 12 , 2010

The Edgeworth Box

2 person, 2 good, pure exchange economy

Fixed positive quantities of X and Y, and two households, 1 and 2.

Household 1 is endowed with $\bar{X}^1$ of good X and $\bar{Y}^1$ of good Y, utility function $U^1(X^1, Y^1)$. Household 2 is endowed with $\bar{X}^2$ of good X and $\bar{Y}^2$ of good Y, utility function $U^2(X^2, Y^2)$

$X^1 + X^2 = \bar{X}^1 + \bar{X}^2 \equiv \bar{X}$,
$Y^1 + Y^2 = \bar{Y}^1 + \bar{Y}^2 \equiv \bar{Y}$.

Each point in the Edgeworth box represents an attainable choice of X^1 and X^2 , Y^1 and Y^2 .
1's origin is at the southwest corner; 1's consumption increases as the allocation point moves in a northeast direction; 2's increases as the allocation point moves in a southwest direction. Superimpose indifference curves on the Edgeworth Box.

Competitive Equilibrium

(p^o_x , p^o_y) so that (X^{o1}, Y^{o1}) maximizes $U^1(X^1, Y^1)$ subject to $(p^o_x , p^o_y)\cdot(X^1, Y^1) \le (p^o_x , p^o_y)\cdot(\bar{X}^1, \bar{Y}^1)$ and

(X^{o2}, Y^{o2}) maximizes $U^2(X^2, Y^2)$ subject to $(p^o_x , p^o_y)\cdot(X^1, Y^1) \le (p^o_x , p^o_y)\cdot(\bar{X}^2, \bar{Y}^2)$, and

$(X^{o1}, Y^{o1}) + (X^{o2}, Y^{o2}) = (\bar{X}^1, \bar{Y}^1) + (\bar{X}^2, \bar{Y}^2)$

or $(X^{o1}, Y^{o1}) + (X^{o2}, Y^{o2}) \le (\bar{X}^1, \bar{Y}^1) + (\bar{X}^2, \bar{Y}^2)$, where the inequality holds co-ordinatewise and any good for which there is a strict inequality has a price of 0.

Pareto efficiency:

An allocation is Pareto efficient if all of the opportunities for mutually desirable reallocation have been fully used. The allocation is Pareto efficient if there is no available reallocation that can improve the utility level of one household while not reducing the utility of any household.

Tangency of 1 and 2's indifference curves : Pareto efficient allocations.

Pareto efficient allocation:

$(X^{o1}, Y^{o1}), (X^{o2}, Y^{o2})$ maximizes

$U^1(X^1, Y^1)$ subject to

$U^2(X^2, Y^2) \geq U^{o2}$ (typically equality will hold and $U^{o2}=U^2(X^{o2}, Y^{o2})$) and subject to the resource constraints

$$X^1 + X^2 = \bar{X}^1 + \bar{X}^2 \equiv \bar{X}$$

$$Y^1 + Y^2 = \bar{Y}^1 + \bar{Y}^2 \equiv \bar{Y} \quad .$$

Equivalently, $X^2 = \bar{X} - X^1$, $Y^2 = \bar{Y} - Y^1$

<u>Solving for Pareto efficiency</u> (Assuming differentiability and an interior solution):

Lagrangian

$$L \equiv U^1(X^1, Y^1) + \lambda[U^2(\bar{X} - X^1, \bar{Y} - Y^1) - U^{o2}]$$

$$\frac{\partial L}{\partial X^1} = \frac{\partial U^1}{\partial X^1} - \lambda \frac{\partial U^2}{\partial X^2} = 0$$

$$\frac{\partial L}{\partial Y^1} = \frac{\partial U^1}{\partial Y^1} - \lambda \frac{\partial U^2}{\partial Y^2} = 0$$

$$\frac{\partial L}{\partial \lambda} = U^2(X^2, Y^2) - U^{o2} = 0$$

This gives us then the condition

$$MRS^1_{xy} = \frac{\frac{\partial U^1}{\partial X^1}}{\frac{\partial U^1}{\partial Y^1}} = \frac{\frac{\partial U^2}{\partial X^2}}{\frac{\partial U^2}{\partial Y^2}} = MRS^2_{xy} \text{ or equivalently}$$

$$MRS^1_{xy} = \frac{\partial Y^1}{\partial X^1}\Big|_{U^1=\text{constant}} = \frac{\partial Y^2}{\partial X^2}\Big|_{U^2=\text{constant}} = MRS^2_{xy}$$

Pareto efficient allocation in the Edgeworth box: the slope of 2's indifference curve at an efficient allocation will equal the slope of 1's indifference curve; the points of tangency of the two curves.

contract curve = individually rational Pareto efficient points

Market allocation

p^x , p^y

Household 1:Choose X^1, Y^1, to maximize $U^1(X^1,Y^1)$ subject to
$p^xX^1 + p^yY^1 = p^x \bar{X}^1 + p^y\bar{Y}^1 = B^1$
budget constraint is a straight line passing through the endowment point $(\bar{X}^1, \bar{Y}^1)$ with slope $-\frac{p^x}{p^y}$.

Lagrangian

$L = U^1(X^1,Y^1) - \lambda [p^xX^1 + p^yY^1 - B^1]$

$$\frac{\partial L}{\partial X} = \frac{\partial U^1}{\partial X^1} - \lambda p^x = 0$$

$$\frac{\partial L}{\partial Y} = \frac{\partial U^1}{\partial Y^1} - \lambda p^y = 0$$

Therefore, at the utility optimum subject to budget constraint we have

$$MRS^1_{xy} = \frac{\frac{\partial U^1}{\partial X^1}}{\frac{\partial U^1}{\partial Y^1}} = \frac{p^x}{p^y}$$; Similarly for household 2,

$$MRS^2_{xy} = \frac{\frac{\partial U^2}{\partial X^2}}{\frac{\partial U^2}{\partial Y^2}} = \frac{p^x}{p^y} \quad .$$

Equilibrium prices: p^{*x} and p^{*y} so that

$X^{*1} + X^{*2} = \bar{X}^1 + \bar{X}^2 \equiv \bar{X}$

$Y^{*1} + Y^{*2} = \bar{Y}^1 + \bar{Y}^2 \equiv \bar{Y}$,

(market clearing)

where X^{*i} and Y^{*i} , i =1, 2, are utility maximizing mix of X and Y at prices p^{*x} and p^{*y} .

$$-\frac{\partial Y^1}{\partial X^1}\Big|_{U^1=U^{1*}} = \frac{\frac{\partial U^1}{\partial X^1}}{\frac{\partial U^1}{\partial Y^1}} = \frac{p^x}{p^y}$$

$$\frac{p^x}{p^y} = \frac{\frac{\partial U^2}{\partial X^2}}{\frac{\partial U^2}{\partial Y^2}} = -\frac{\partial Y^2}{\partial X^2}\Big|_{U^2=U^{2*}}$$

The price system decentralizes the efficient allocation decision.

Set Theory

Logical Inference

Let A and B be two logical conditions, like A="it's sunny today" and B="the light outside is very bright"

A $\Rightarrow$ B

A implies B, if A then B

A $\Leftrightarrow$ B

A if and only if B, A implies B and B implies A, A and B are equivalent conditions

Definition of a Set

{ }

{x | x has property P}

{1, 2, ..., 9, 10} = { x | x is an integer, 1≤ x ≤ 10 }.

Elements of a set

x ∈ A ; $y \notin A$

x ≠ { x }

x ∈ { x }

ϕ ≡ the empty set (≡ null set), the set with no elements.

Subsets

$A \subset B$ or $A \subseteq B$ if x ∈ A ⇒ x ∈ B

$A \subset A$ and $\phi \subset A$.

Set Equality

A = B if A and B have precisely the same elements

A = B if and only if $A \subset B$ and $B \subset A$.

Set Union

$A \cup B$

$A \cup B = \{x;|;x \in A \text{ or } x \in B\}$ ('or' includes 'and')

Set Intersection

$\cap$

$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

If $A \cap B = \phi$ we say that A and B are disjoint.

Theorem 6.1: Let A, B, C be sets,

a. $A \cap A = A,\ A \cup A = A$ (idempotency)

b. $A \cap B = B \cap A,\ A \cup B = B \cup A$ (commutativity)

c. $A \cap (B \cap C) = (A \cap B) \cap C$ (associativity)

$A \cup (B \cup C) = (A \cup B) \cup C$

d. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (distributivity)

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Complementation (set subtraction)

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$A \backslash B = \{x \mid x \in A,\ x \notin B\}$

Cartesian Product

ordered pairs

$A\ x\ B = \{(x,\ y) \mid x \in A,\ y \in B\}$.

Note: If x ≠ y, then (x, y) ≠ (y, x) .

R = The set of real numbers

$\mathbf{R}^N$ = N-fold Cartesian product of R with itself.

$\mathbf{R}^N$ = R x R x R x ... x R, where the product is taken N times.

The order of elements in the ordered N-tuple (x, y, ...) is essential. If $x \neq y,\ (x, y, \ldots) \neq (y, x, \ldots)$.

$\mathbf{R}^N$, Real N-dimensional Euclidean space

Read Starr's *General Equilibrium Theory*, Chapter 7.

R^2 = plane

R^3 = 3-dimensional space

R^N = N-dimensional Euclidean space

Definition of R:

R = the real line

$\pm\infty \notin$ R

+, - , ×, ÷

closed interval : $[a, b] \equiv \{x | x \in R, a \le x \le b\}$.

R is *complete*. Nested intervals property: Let $x^\nu < y^\nu$ and $[x^{\nu+1}, y^{\nu+1}] \subseteq [x^\nu, y^\nu]$, $\nu = 1, 2, 3, \ldots$. Then there is $z \in R$ so that $z \in [x^\nu, y^\nu]$, for all ν .

R^N = N-fold Cartesian product of R.

$x \in R^N$, $x = (x_1, x_2, \ldots, x_N)$

x_i is the ith co-ordinate of x.

x = point (or *vector*) in R^N

Algebra of elements of R^N

$x + y = (x_1 + y_1,\ x_2 + y_2,\ \ldots,\ x_N + y_N)$

0 = (0, 0, 0, ..., 0) , the origin in N-space

$x - y \equiv x + (-y) = (x_1 - y_1, x_2 - y_2, \ldots, x_N - y_N)$

$t \in R,\ x \in R^N$, then $tx \equiv (tx_1,\ tx_2,\ \ldots,\ tx_N)$.

$x,\ y \in R^N,\quad x \cdot y = \sum_{i=1}^{N} x_i y_i$. If $p \in R^N$ is a price vector and $y \in R^N$ is an economic action, then $p \cdot y = \sum_{n=1}^{N} p_n y_n$ is the value of the action y at prices p.

Norm in R^N, the measure of distance

$$|x| \equiv \|x\| \equiv \sqrt{x \cdot x} \equiv \sqrt{\sum_{i=1}^{N} x_i^2} \quad .$$

Let $x,\ y \in R^N$. The distance between x and y is $\|x - y\|$.

$| x - y | = \sqrt{\Sigma_i (x_i - y_i)^2}$.

$\|x - y\| \ge 0$ all $x,\ y \in R^N$

$| x - y | = 0$ if and only if $x = y$.

Limits of Sequences

x^ν , $\nu = 1, 2, 3, \ldots$,

Example: $x^\nu = 1/\nu$. 1, 1/2, 1/3, 1/4, 1/5, ... $x^\nu \to 0$.

Formally, let $x^i \in R,\ i = 1,\ 2,\ \ldots$. Definition: We say $x^i \to x^0$ if for any $\varepsilon > 0$, there is $q(\varepsilon)$ so that for all $q' > q(\varepsilon)$, $|x^{q'} - x^0| < \varepsilon$.

So in the example $x^\nu = 1/\nu$, $q(\varepsilon) = 1/\varepsilon$

Let $x^i \in R^N,\ i = 1,\ 2,\ \ldots$. We say that $x^i \to x^0$ if for each co-ordinate $n = 1,\ 2,\ \ldots,\ N,\ x_n^i \to x_n^0$.

Theorem 7.1: Let $x^i \in R^N,\ i = 1,\ 2,\ \ldots$. Then $x^i \to x^0$ if and only if for any ε there is $q(\varepsilon)$ such that for all $q' > q(\varepsilon)$, $\left\|x^{q'} - x^0\right\| < \varepsilon$.

x^o is a *cluster point* of $S \subseteq \mathbf{R}^N$ if there is a sequence $x^\nu \in R^N$ so that $x^\nu \to x^o$.

Open Sets

Let $X \subset R^N$; X is *open* if for every $x \in X$ there is an $\varepsilon > 0$ so that $\|x - y\| < \varepsilon$ implies $y \in X$.

Open interval in R: $(a, b) = \{ x \mid x \in R, a < x < b\}$

ϕ and R^N are open.

Closed Sets

Example: Problem - Choose a point x in the closed interval [a, b] (where $0 < a < b$) to maximize x^2. Solution: x = b.

Problem - Choose a point x in the open interval (a, b) to maximize x^2. There is no solution in (a, b) since $b \notin (a, b)$.

A set is closed if it contains all of its cluster points.

Definition: Let $X \subset R^N$. X is said to be a **closed** set if for every sequence x^ν, $\nu = 1, 2, 3, \ldots$, satisfying,

(i) $x^\nu \in X$, and

(ii) $x^\nu \to x^0$,

it follows that $x^0 \in X$.

Examples: A closed interval in R, [a, b] is closed

A closed ball in R^N of radius r, centered at $c \in R^N$, $\{x \in R^N \mid |x-c| \le r\}$ is a closed set.

A line in R^N is a closed set

But a set may be neither open nor closed (for example the sequence $\{1/\nu\}$, $\nu=1, 2, 3, 4, \ldots$ is not closed in R, since 0 is a limit point of the sequence but is not an element of the sequence; it is not open since it consists of isolated points).

Note: Closed and open are not antonyms among sets. ϕ and R^N are each both closed and open.

Let $X \subseteq R^N$. The closure of X is defined as

$\bar{X} \equiv \{ y \mid \text{there is } x^\nu \in X, \nu = 1, 2, 3, \ldots, \text{ so that } x^\nu \to y \}$.

For example the closure of the sequence in R, $\{1/\nu \mid \nu=1, 2, 3, 4, \ldots\}$ is

$\{0\}\cup\{1/\nu \mid \nu=1, 2, 3, 4, \ldots\}$.

Concept of Proof by contradiction: Suppose we want to show that **A** $\Rightarrow$ B. Ordinarily, we'd like to prove this directly. But it may be easier to show that [not (**A** $\Rightarrow$ B)] is false. How? Show that [A & (not B)] leads to a contradiction. A: x = 1, B:x+3=4. Then [A & (not B)] leads to the conclusion that $1+3\neq4$ or equivalently $1\neq1$, a contradiction. Hence [A & (not B)] must fail so A$\Rightarrow$ B. (Yes, it does feel backwards, like your pocket is being picked, but it works).

Theorem 7.2: Let $X \subset R^N$. X is closed if $R^N \setminus X$ is open.
Proof: Suppose $R^N \setminus X$ is open. We must show that X is closed. If $X=R^N$ the result is trivially satisfied. For $X\neq R^N$, let $x^\nu \in X$, $x^\nu \to x^o$. We must show that $x^o \in X$ if $R^N \setminus X$ is open. Proof by contradiction. Suppose not. Then $x^o \in R^N \setminus X$. But $R^N \setminus X$ is open. Thus there is an ε neighborhood about x^o entirely contained in $R^N \setminus X$. But then for ν large, $x^\nu \in R^N \setminus X$, a contradiction. Therefore $x^o \in X$ and X is closed. QED

Theorem 7.3: 1. $X \subset \bar{X}$
2. $X = \bar{X}$ if and only if X is closed.

Bounded Sets

Def: $K(k) = \{x \mid x \in R^N,\ |x_i| \le k,\ i = 1,\ 2,\ \ldots,\ N\}$ = cube of side 2k (centered at the origin).

Def: $X \subset R^N$. X is *bounded* if there is $k \in R$ so that $X \subset K(k)$.

Compact Sets

THE IDEA OF COMPACTNESS IS ESSENTIAL!

Def: $X \subset R^N$. X is *compact* if X is closed and bounded.

Finite subcover property: An open covering of X is a collection of open sets so that X is contained in the union of the collection. It is a property of compact X that for every open covering there is a finite subset of the open covering whose union also contains X. That is, every open covering of a compact set has a finite subcover.

Boundary, Interior, etc.

$X \subset R^N$, Interior of X = $\{y \mid y \in X$, there is $\varepsilon > 0$ so that $\|x - y\| < \varepsilon$ implies $x \in X\}$

Boundary $X \equiv \bar{X} \setminus$Interior X

Set Summation in R^N

Let $A \subseteq R^N$, $B \subseteq R^N$. Then

$A + B \equiv \{ x \mid x = a + b, a \in A, b \in B \}$.

The Bolzano-Weierstrass Theorem, Completeness of R^N .

Theorem 7.4 (Nested Intervals Theorem): By an interval in R^N , we mean a set I of the form $I = \{(x_1,\ x_2,\ \ldots,\ x_N) |\ a_1 \le x_1 \le b_1\ \ , a_2 \le x_2 \le b_2,\ \ldots,\ a_N \le x_N \le b_N,\ a_i,\ b_i \in R\}$. Consider a sequence of nonempty closed intervals I_k such that

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots \supseteq I_k \supseteq \ldots \ .$$

Then there is a point in R^N contained in all the intervals. That is, $\exists x^o \in \bigcap_{i=1}^{\infty} I_i$ and therefore $\bigcap_{i=1}^{\infty} I_i \neq \phi$; the intersection is nonempty.

Proof: Follows from the completeness of the reals, the nested intervals property on R.

Corollary (Bolzano-Weierstrass theorem for sequences): Let x^i , i = 1, 2, 3, ... be a bounded sequence in R^N . Then x^i contains a convergent subsequence.

Proof 2 cases: x^i assumes a finite number of values, x^i assumes an infinite number of values.

It follows from the Bolzano-Weierstrass Theorem for sequences and the definition of compactness that an infinite sequence on a compact set has a convergent subsequence whose limit is in the compact set.